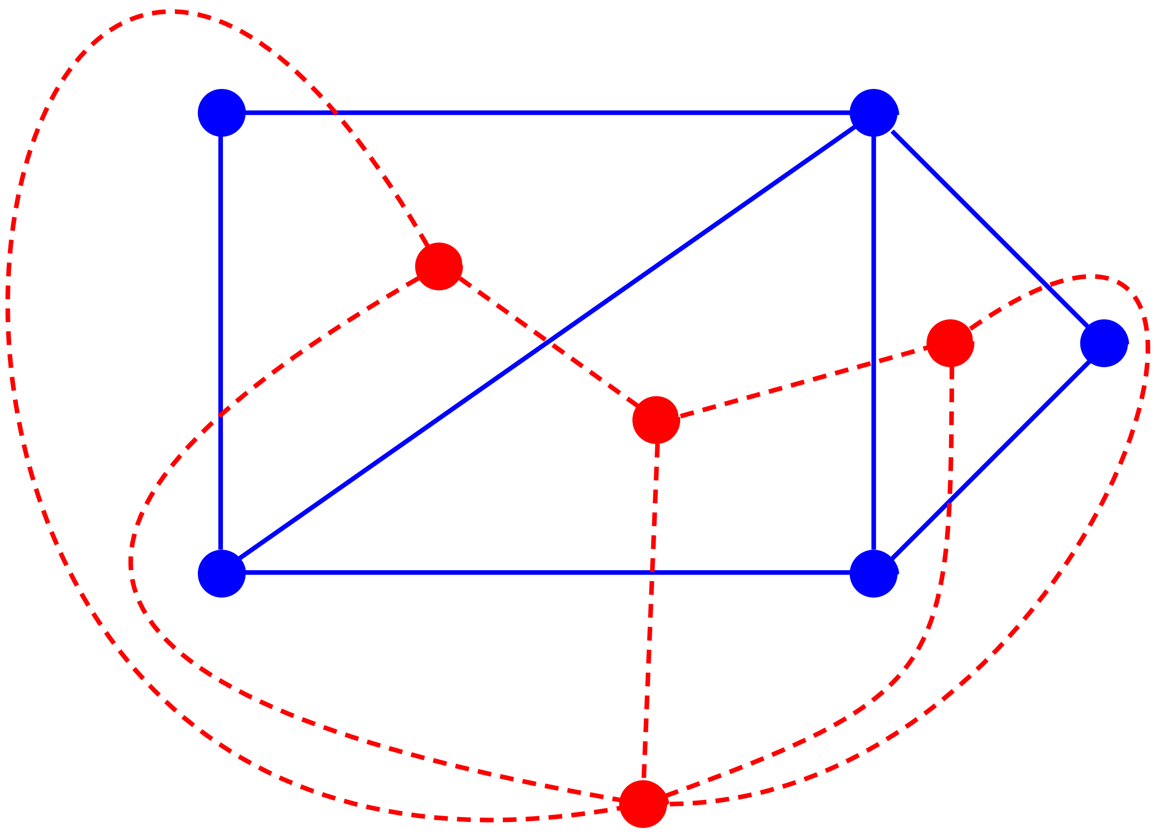
Duality

CSC 282

Duality

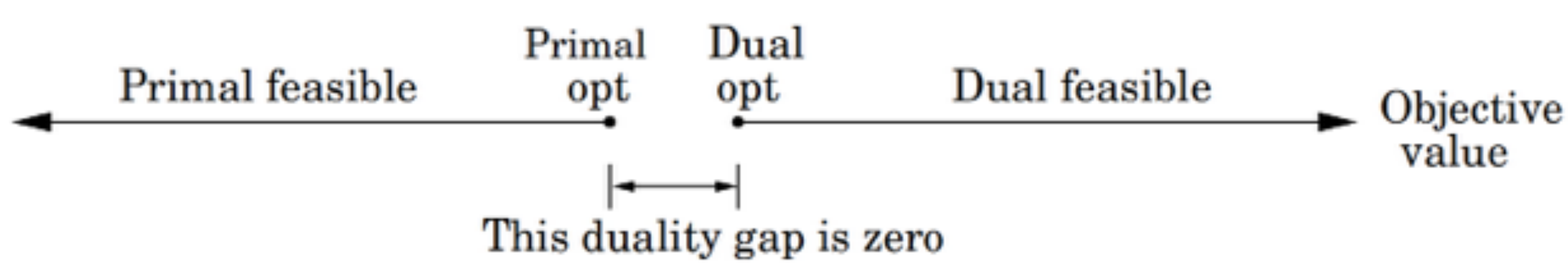
- Change of representation by exchanging parts of one type with parts of another part
- Example: Dual of a planar graph: exchange vertices and faces



Duality in Linear Programming

- Exchange variables and constraints
- Exchange maximization and minimization
- Exchange constraints and objective function
- Feasible solutions to original problem coincide with feasible solutions to the dual at the point of optimality

Figure 7.9 By design, dual feasible values $\geq$ primal feasible values. The duality theorem tells us that moreover their optima coincide.



Intuition

- The primal (original) problem asks us to maximize some linear expression
- We can find a linear combination of the constraints that puts an upper bound on that expression
- The best (tightest) such upper bound is exactly at the optima of the primal problem

Example

	Primal	Dual
	$\max\ x_1 + 6x_2$	$\min\ 200y_1 + 300y_2 + 400y_3$
y_1	$x_1 \leq 200$	
y_2	$x_2 \leq 300$	$y_1 + y_3 \geq 1$
y_3	$x_1 + x_2 \leq 400$	$y_2 + y_3 \geq 6$
	$x_1, x_2 \geq 0$	$y_1, y_2, y_3 \geq 0$

$$x_1 + 6x_2 \quad \leq$$

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \quad \leq \quad 200y_1 + 300y_2 + 400y_3$$

Matrix Form

A linear function like $x_1 + 6x_2$ can be written as the dot product of two vectors

$$\mathbf{c} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

denoted $\mathbf{c} \cdot \mathbf{x}$ or $\mathbf{c}^T\mathbf{x}$. Similarly, linear constraints can be compiled into matrix-vector form:

$$\begin{array}{rcl} x_1 & \leq & 200 \\ x_2 & \leq & 300 \\ x_1 + x_2 & \leq & 400 \end{array} \quad \Longrightarrow \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} \underbrace{\leq}_{\leq} \underbrace{\begin{pmatrix} 200 \\ 300 \\ 400 \end{pmatrix}}_{\mathbf{b}}.$$

Primal LP:

$$\max\ \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

Dual LP:

$$\min\ \mathbf{y}^T\mathbf{b}$$
$$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$$
$$\mathbf{y} \geq 0$$

solve_lp dual1.lp

Why Care About Duality?

- Dual problem might be easier to solve than primal
- If solver cannot be run all the way to completion
  - Primal provides a lower bound on best solution
  - Dual provides an upper bound on best solution
- Strong duality: for LP, if you can find any pair of feasible solutions for the primal and dual with the same objective value, you are done!
- Duals can be defined for harder classes of problems, e.g. mixed-integer programming
  - But in general only weak duality holds

Problems