CSC242: Intro to AI

Lecture 13

Recap

Elements of First-Order Logic

- Objects (in the world)
- Relations among (tuples of) objects
- Mappings from (tuples of) objects to objects (i.e., objects identified in terms of other objects)
- Connectives
- Variables and Quantifiers

Syntax of First-Order Logic

- Constants } Terms
- Functional expressions
- Atomic sentences
- Connectives
- Variables and Quantifiers

Interpretation

- Domain of objects D
- Mapping from constant symbols to objects:
  $$I(c) \in D$$
- Mapping from predicate symbols to relations: $I(p) \subseteq D^n \quad \text{for } p \text{ of arity } n$
- Mapping from function symbols to (total) functions: $I(f) : D^n \rightarrow D \quad \text{for } f \text{ of arity } n$

Semantics of Unquantified Sentences

$P(\langle x_1, \ldots, x_n \rangle)$ is true in interpetation I if

$$\langle I(x_1), \ldots, I(x_n) \rangle \in I(P)$$

Connectives: use truth tables (as for PL)

Extended Interpretation

- Adds mapping from variables to objects:

$$I(v) \in D$$

Semantics of Quantified Sentences

$\forall x\ \alpha$ is true in interpetation I if

$I'(\alpha)$ is true in every extended interpetation I'

$\exists x\ \alpha$ is true in interpetation I if

$I'(\alpha)$ is true in some extended interpetation I'

Entailment

- If α entails β: $\alpha \models \beta$
  - Whenever α is true, so is β
  - Every model of α is a model of β
  - Models(α) ⊆ Models(β)
  - α is at least as strong an assertion as β
    - Rules out no fewer possible worlds

Computing Entailment

- Number of models (probably) unbounded
  - And anyway hard to evaluate truth in a model
- Can't do model checking
- Look for inference rules, do theorem proving

First-Order Inference

- $\forall x$ King(x) $\Rightarrow$ Evil(x)
- King(J)

- $\forall x\ King(x) \Rightarrow Evil(x)$
- King(J)
- Evil(J)

"Modus Ponens"

Universal Instantiation

$$\frac{\forall x\, P(x)}{P(a), P(b), P(c), P(f(a)), P(f(b)), P(g(a)), \ldots}$$

Propositionalization

$\forall x$ King(x) $\Rightarrow$ Evil(x)

King(R) $\Rightarrow$ Evil(R)

King(J) $\Rightarrow$ Evil(J)

King(Father(R)) $\Rightarrow$ Evil(Father(R))

...

Propositionalization

- Convert FOL sentences to PL sentences and do PL inference on them
- Seems kind of indirect
- Technical problem enumerating all the ground terms (could be infinite; see book)

- $\forall x\ King(x) \Rightarrow Evil(x)$
- $King(J)$

- $\forall x\ King(x) \Rightarrow Evil(x)$
- $King(J)$

- King(J) $\Rightarrow$ Evil(J)
- King(J)

- $King(J) \Rightarrow Evil(J)$
- $King(J)$
- $Evil(J)$

- $\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
- King(J)
- Greedy(J), Greedy(R)

- $\forall x$ King(x) $\wedge$ Greedy(x) $\Rightarrow$ Evil(x)
- King(J)
- Greedy(J), Greedy(R)

- $\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
- King(J)
- Greedy(J), Greedy(R)

- $\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$
- King(J)
- Greedy(J), Greedy(R)
- Evil(J)

Substitution

- Replacement (binding) of variables by terms
  - { x/J }
  - { x/R, y/Father(J) }

- $\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$
- King(J) { x/J }
- Greedy(J), Greedy(R)

If there is some substitution that makes the premises true, then the conclusion with the same substitution is also true

Generalized Modus Ponens

$$\frac{p_1', p_2', \cdots, p_n',\ (p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow q)}{Subst(\Theta, q)}$$

p_i, p_i', q are atomic sentences

Θ is a substitution such that:

$$Subst(\Theta, p_i') = Subst(\Theta, p_i)$$

Lifted Inference Rule

- Inference rule lifted from ground (variable-free) propositional logic to first-order logic
- “The key advantage of lifted inference rules over propositionalization is that they make only those substitutions that are required to allow particular inferences to succeed”

Unit Resolution

Clause

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k}$$

Unit Clause

$l_l, \ldots, l_k$ are <u>literals</u>

li and *m* are <u>complementary</u>

Positive literal: P

Negative literal: ¬P

Complementary literals

Resolution

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_j \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \cdots \vee m_n}$$

$l_1, ..., l_k$, $m_1, ..., m_n$ are literals

l_i and m_j are complementary

Technical note: Resulting clause must be factored to contain only one copy of each literal.

Resolution

- Sound
- Complete
- Can it be lifted to FOL?

CNF for FOL

- Every sentence of first-order logic can be converted into an inferentially equivalent sentence in conjunctive normal form (CNF)

CNF for FOL

- Eliminate implications
- Move negation inwards
- Standardize variables apart
- Skolemize existential variables
- Drop universal variables
- Distribute $\vee$ over $\wedge$

Example

- $\forall x\ (Person(x) \Rightarrow \neg \forall y\ (\neg Loves(x,y) \vee \neg Person(y))$
- $\forall x\ (\neg Person(x) \vee \neg \forall y\ (\neg Loves(x,y) \vee \neg Person(y))$
- $\forall x\ (\neg Person(x) \vee \exists y\ \neg\ (\neg Loves(x,y) \vee \neg Person(y))$
- $\forall x\ (\neg Person(x) \vee \exists y\ (\neg\neg\ Loves(x,y) \wedge \neg\neg\ Person(y))$
- $\forall x(\neg Person(x) \vee \exists y\ (Loves(x,y) \wedge Person(y))$
- $\forall x\ (\neg Person(x) \vee (Loves(x,f(x)) \wedge Person(f(x)))$
- $(\neg Person(x) \vee (Loves(x,f(x)) \wedge Person(f(x)))$
- $(\neg Person(x) \vee Loves(x,f(x))) \wedge (\neg Person(x) \vee Person(f(x)))$

CNF for FOL

- Every sentence of first-order logic can be converted into an inferentially equivalent sentence in conjunctive normal form (CNF)
- Not logically equivalent
  - Why? Skolemization
- Inferentially equivalent: (un)satisfiable iff original sentence is (un)satisfiable

Entailment by Unsatisfiability

- To prove $\alpha \models \beta$, prove $\alpha \cup \neg\beta$ is unsatisfiable (has no models)
- Why does this work?
  - All models of α are models of β
    - So, no models of α are models of $\neg\beta$
  - Conversely: suppose α has model M
    - α must make $\neg\beta$ false, so it makes β true

Refutation Proofs

- To prove $\alpha \models \beta$, prove $\alpha \cup \neg\beta \vdash$ FALSE
- FALSE = some formula that clearly is unsatisfiable (has no models)
- CNF: an “empty” clause means FALSE
  - Empty clause: disjunction of NO literals
  - ( )

Example Refutation Proof

- Given: (Floats⇒Witch), Floats
  - Prove: Witch
- Convert to CNF and add negated goal

  (¬Floats ∨ Witch), (Floats), (¬Witch)
- Modus Ponens:
  - (¬Floats ∨ Witch), (Floats) ⊢ (Witch)
  - (Witch), (¬Witch) ⊢ ( )

Resolution

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_j \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \cdots \vee m_n}$$

$l_1, ..., l_k$, $m_1, ..., m_n$ are literals

li and m_j are complementary

Technical note: Resulting clause must be factored to contain only one copy of each literal.

- $\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$
- King(J)
- $\forall y\ Greedy(y)$
- What substation is needed? { x/J, y/J }

Unification

$$Unify(p, q) = \Theta \text{ where } Subst(\Theta, p) = Subst(\Theta, q)$$

- Knows(John,x) and Knows(John,Jane)
  - { x/Jane }
- Knows(John,x) and Knows(y, Bill)
  - {x/Bill, y/John }
- Knows(John,x) and Knows(y, Mother(y))
  - { y/John, x/Mother(John) }
- Knows(John,x) and Knows(x,Elizabeth)
  - Fails

Unification

$$Unify(p, q) = \Theta \text{ where } Subst(\Theta, p) = Subst(\Theta, q)$$

- Variables need to be standardized apart
- Occurs check
- Most general unifier: places the fewest restrictions on the variables
  - Is unique
  - AIMA Fig 9.1

FOL Resolution

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_j \vee \cdots \vee m_n}{Subst(\Theta, l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \cdots \vee m_n)}$$

$$\Theta = \ Unify(l_i, \neg m_j)$$

Proof by Resolution

- Convert KB to CNF
- Convert ¬α to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all its missiles were sold to it by Colonel West, who is American.

To prove: West is a criminal

It is a crime for an American to sell weapons to hostile nations.

$$\forall x, y, z\ American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$$

Nono has some missiles.

$$\exists x\ Owns(Nono, x) \land Missile(x)$$

All Nono's missiles were sold to it by Colonel West.

$$\forall x\ Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$$

Missiles are weapons.

$$\forall x\ Missile(x) \Rightarrow Weapon(x)$$

An enemy of America counts as "hostile."

$$\forall x\ Enemy(x, America) \Rightarrow Hostile(x)$$

West is an American.

$$American(West)$$

Nono is an enemy of America.

$$Enemy(Nono, America)$$

$$\forall x, y, z\ American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$$

$$\exists x\ Owns(Nono, x) \land Missile(x)$$

$$\forall x\ Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$$

$$\forall x\ Missile(x) \Rightarrow Weapon(x)$$

$$\forall x\ Enemy(x, America) \Rightarrow Hostile(x)$$

$$American(West)$$

$$Enemy(Nono, America)$$

Proof by Resolution

- Convert KB to CNF
- Convert ¬α to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause

Convert to CNF

- Eliminate implications
- Move negation inwards
- Standardize variables apart
- Skolemize existential variables
- Drop universal variables
- Distribute $\vee$ over $\wedge$

$$\forall x, y, z\ American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$$

Eliminate implications:

$$\forall x, y, z\ \neg[American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z)] \vee Criminal(x)$$

Move negation inwards (DeMorgan's Laws)

$$\forall x, y, z\ [\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z)] \vee Criminal(x)$$

Standardize variables apart

Skolemize existential variables

Drop universal quantifiers

$$\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z) \vee Criminal(x)$$

Distribute $\vee$ over $\wedge$

$$\forall x, y, z\ American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$$

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$$

$$\forall x, y, z \; American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$$

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$$

$$\exists x \; Owns(Nono, x) \land Missile(x)$$

$$Owns(Nono, M_1), Missile(M_1)$$

$$\forall x \; Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$$

$$\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$$

$$\forall x \; Missile(x) \Rightarrow Weapon(x)$$

$$\neg Missile(x) \lor Weapon(x)$$

$$\forall x \; Enemy(x, America) \Rightarrow Hostile(x)$$

$$\neg Enemy(x, America) \lor Hostile(x)$$

$$American(West) \qquad Enemy(Nono, America)$$

$$American(West) \qquad Enemy(Nono, America)$$

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$$

$$Owns(Nono, M_1), Missile(M_1)$$

$$\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$$

$$\neg Missile(x) \lor Weapon(x)$$

$$\neg Enemy(x, America) \lor Hostile(x)$$

$$American(West)$$

$$Enemy(Nono, America)$$

Proof by Resolution

- Convert KB to CNF
- Convert ¬α to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause

“West is a criminal”

$$Criminal(West)$$

Negate and convert to CNF

$$\neg Criminal(West)$$

$$\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z) \vee Criminal(x)$$

$$Owns(Nono, M_1), Missile(M_1)$$

$$\neg Missile(x) \vee \neg Owns(Nono, x) \vee Sells(West, x, Nono)$$

$$\neg Missile(x) \vee Weapon(x)$$

$$\neg Enemy(x, America) \vee Hostile(x)$$

$$American(West)$$

$$Enemy(Nono, America)$$

$$\neg Criminal(West)$$

Proof by Resolution

- Convert KB to CNF
- Convert ¬α to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$$

$$Owns(Nono, M_1), Missile(M_1)$$

$$\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$$

$$\neg Missile(x) \lor Weapon(x)$$

$$\neg Enemy(x, America) \lor Hostile(x)$$

$$American(West)$$

$$Enemy(Nono, America)$$

$$\neg Criminal(West)$$

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor \boxed{Criminal(x)}$$

$$Owns(Nono, M_1), Missile(M_1)$$

$$\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$$

$$\neg Missile(x) \lor Weapon(x)$$

$$\neg Enemy(x, America) \lor Hostile(x)$$

$$American(West)$$

{ x/West }

$$Enemy(Nono, America)$$

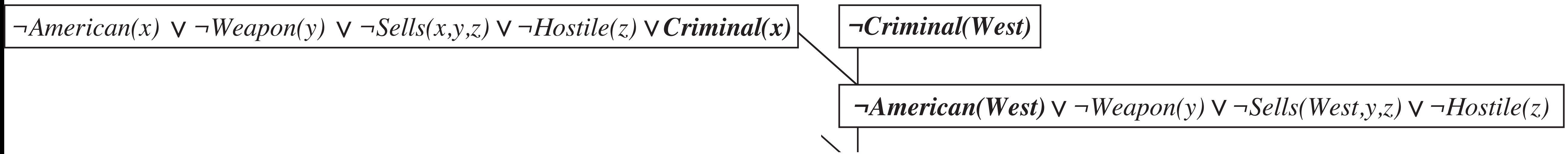
$$\boxed{\neg Criminal(West)}$$

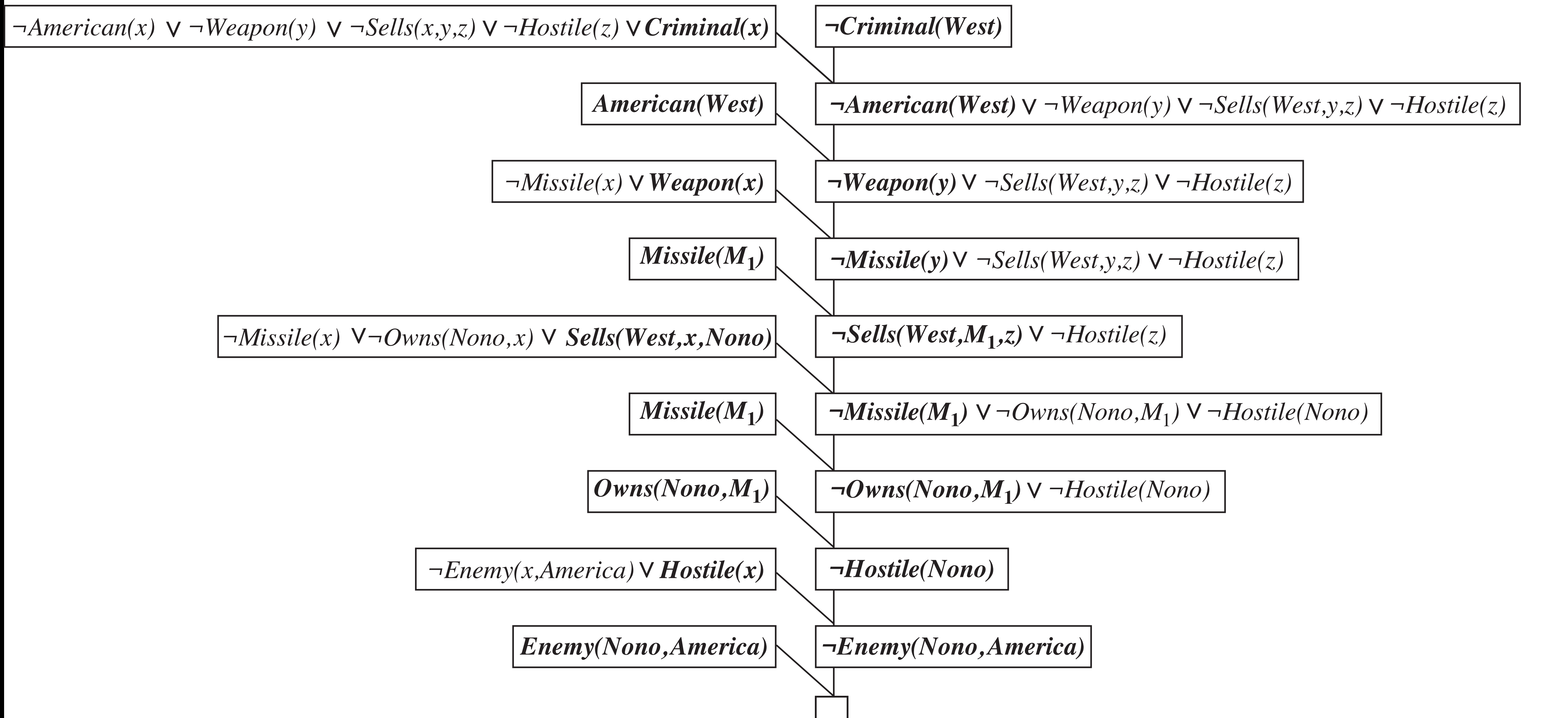
$$\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z) \vee Criminal(x)$$

$$\neg Criminal(West)$$

{ x/West }

$$\neg American(West) \vee \neg Weapon(y) \vee \neg Sells(West, y, z) \vee \neg Hostile(z)$$


¬American(x) ∨ ¬Weapon(y) ∨ ¬Sells(x,y,z) ∨ ¬Hostile(z) ∨ Criminal(x)
¬Criminal(West)
¬American(West) ∨ ¬Weapon(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)


¬American(x) ∨ ¬Weapon(y) ∨ ¬Sells(x,y,z) ∨ ¬Hostile(z) ∨ Criminal(x)
¬Criminal(West)
American(West)
¬American(West) ∨ ¬Weapon(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)
¬Missile(x) ∨ Weapon(x)
¬Weapon(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)
Missile(M1)
¬Missile(y) ∨ ¬Sells(West,y,z) ∨ ¬Hostile(z)
¬Missile(x) ∨ ¬Owns(Nono,x) ∨ Sells(West,x,Nono)
¬Sells(West,M1,z) ∨ ¬Hostile(z)
Missile(M1)
¬Missile(M1) ∨ ¬Owns(Nono,M1) ∨ ¬Hostile(Nono)
Owns(Nono,M1)
¬Owns(Nono,M1) ∨ ¬Hostile(Nono)
¬Enemy(x,America) ∨ Hostile(x)
¬Hostile(Nono)
Enemy(Nono,America)
¬Enemy(Nono,America)

Resolution Proof

Since $KB \cup \{\neg Criminal(West)\}$ is unsatisfiable,

$$KB \models Criminal(West)$$

Special Case: Forward Chaining

- Knowledge base of definite clauses
  - Implications with single literal conclusion
- Starting from known facts, trigger rules whose premises are satisfied
  - Using substitution to match
- Add their conclusions to the KB

Special Case: Backward Chaining

- Work backward from the goal, chaining through rules to find known facts that support the proof
  - Allow substitutions when matching facts and rules
- DFS => incomplete
- The basis of logic programming (Prolog)

FOL Inference

- Semantics of first-order sentences
- Entailment: same as PL!
- Lifted inference rules
- Resolution (first-order CNF, unification)
  - Proof by contradiction
- Forward and backward chaining

Othello Due 11:49pm
Tonight!
Homework 3 (Logic) posted
Tonight!